

hep-th/0403074  
CALT-68-2478

# Gravitational Perturbations of a Six Dimensional Self-Tuning Model

**M.L. Graesser, J. E. Kile, P. Wang**

*California Institute of Technology, 452-48, Pasadena, CA 91125*

## Abstract

We investigate gravitational perturbations in a compact six-dimensional self-tuning brane model. We specifically look for analytic solutions to the perturbed Einstein equations that correspond in four-dimensions to massless or approximately massless scalars coupled to matter on the brane. The presence of such modes with gravitational couplings would be phenomenologically unacceptable. The most general solution for all such modes is obtained, but it is found that they are all eliminated by the boundary conditions. Our main result is that to linear order in perturbation theory this model does not contain any light scalars. We speculate that this model does not self-tune.

# 1 Introduction

There has been much speculation on the possibility that the Standard Model fields are confined on a four-dimensional brane in a higher-dimensional universe [1]. The usual cosmological constant problem is reformulated in these theories, since in general the cosmological constant in the four dimensional effective theory receives contributions from both bulk physics and from brane physics. The cosmological problem in these models is balancing the bulk terms against the vacuum energy on the brane to produce the very small value seen in nature.

A general class of five-dimensional models [2] were introduced to partially resolve this problem. Instead of canceling bulk terms against brane terms, these models have the interesting feature that flat space solutions always exist for arbitrary values of the tension on the brane. This is a big improvement, since for these class of models there is a hope that the cosmological constant problem could be immune to phase transitions on the brane, although this has not been demonstrated.

It has been recognized that six-dimensional brane world models might be more promising for the cosmological constant problem, due to the co-dimension-two nature of the geometry [3–5]. As might be expected by locality, a three brane with arbitrary tension in six-dimensional flat space does not cause the space to inflate, but just introduces a conical singularity. But as such a model does not lead to four-dimensional Einstein gravity, one must compactify the transverse space.

An interesting co-dimension-two spherical compactification was originally considered by Sundrum [4], and modified by Carroll and Guica [6] and also by Navarro [7]. In their model a fine-tuning of bulk parameters is required to obtain a small four dimensional cosmological constant. What is intriguing though, is that this fine tuning is independent of the tension on the brane. With non-vanishing brane tension the bulk geometry is still locally a sphere, but globally it has a conical deficit angle. Pictorially, a “banana peel” has been removed.

A generic difficulty with the five-dimensional models was that self-tuning required a curvature singularity in the bulk [8]. Resolving the singularity by introducing another brane in the bulk reintroduces a fine-tuning of the brane tensions [9]. Naively the six-dimensional model may be an improvement on the five-dimensional self-tuning models, since here the curvature singularities are conical and perhaps less severe. Still, this may be too much. [10] finds that in geometries with conical singularities it is inconsistent to add anything other than tension to the brane. Since we would like to also have stars and dogs on the brane, this is clearly problematic. It is unclear though whether this is an artifact of treating the brane as infinitely thin <sup>1</sup>, and it would be interesting to see

---

<sup>1</sup>Finite thickness co-dimension-2 models have been discussed in [11].

whether higher dimension operators on the brane could overcome this obstacle<sup>2</sup>.

In this note we will not dispel this concern. Rather, we address an independent issue, which is whether this six-dimensional model leads to a scalar-tensor theory of gravity at low energies. Given the presupposition of a self-tuning mechanism, one might expect such a mode is necessary, for example, to self-tune a change in the brane tension that is much smaller than the compactification scale.

As is well known, gravitational couplings of a scalar admixture to gravity are strongly constrained by measurements [12] of the Nordtvedt effect [13]. Other phenomenological constraints could be obtained from cosmology, for here parameters in the four-dimensional effective theory depend on the brane tension, which probably had a cosmological history [14].

Our main result is that while we do find massless scalars allowed by the bulk equations of motion, these are all eliminated by the boundary conditions. Our analysis can also be extended to exclude light scalars whose mass vanishes in the limit that the tension goes to zero. To linear order in perturbation theory then, this model does not have any phenomenological difficulties of this sort. Our results support the conclusions of [15], but we improve on their work since here we are able to obtain all of our results analytically, without having to resort to a numerical analysis.

If this model does have a self-tuning mechanism, then the absence of any massless scalars does raise a puzzle though, for there is no light scalar to adjust a change in the brane tension. If this model does in fact have a self-tuning mechanism, then our results surprisingly suggest that by default, it is the collective motion of many massive Kaluza-Klein states that is responsible for canceling a change in the tension.

But there is another reason to doubt whether this model has a self-tuning mechanism. For here the deficit angle is an integration parameter that may be chosen to satisfy the boundary conditions after assuming a four-dimensional flat space ansatz. However, one might be worried that this feature may not be sufficient in order to realize the self-tuning. We elaborate on this in Section 4.

The content of this paper is as follows. In section 2, we briefly review the key features of the model. In section 3.1, we determine Einstein's equations to linear order and discuss the appropriate gauge fixing. Here there is a subtlety due to a brane bending mode, and we discuss how we gauge fix this mode. Then we present our solutions for the most general massless scalar modes that could couple to matter on the brane. In section 3.2 we discuss the boundary conditions. After imposing these conditions, none of our zero modes survive. Section 4 reconsiders the self-tuning feature of this model. Section 5 contains some concluding remarks.

---

<sup>2</sup> [10] did find that adding the Gauss-Bonnet operator in the bulk could lead to Einstein gravity with arbitrary sources.

A few notes on notation are in order. We will use  $G_{AB}$  to denote the full 6D metric, Greek indices for the non-compact four dimensions, and Roman letters for components of tensors in the internal directions.

## 2 The Unperturbed Model

The authors of [6] and [7] consider a six dimensional model with two dimensions compact. The bulk geometry has the topology of a sphere with metric

$$ds^2 = r^2 g_{ij} dx^i dx^j = r^2 [d\theta^2 + \beta^2 \sin^2 \theta d\alpha^2] \quad (1)$$

and  $\alpha$  has period  $2\pi$ .

The field content in the bulk is six -dimensional gravity together with a  $U(1)$  gauge field. The field strength for the gauge field is non-vanishing on the sphere,

$$F_{ij} = \sqrt{G^{(2)}} \epsilon_{ij} B, \quad (2)$$

with  $\epsilon_{\theta\phi} \equiv 1$ . The corresponding flux

$$\Phi = \int d^2x \sqrt{G^{(2)}} \epsilon^{ij} F_{ij} \quad (3)$$

is conserved.

There is also a bulk cosmological constant  $\Lambda$ . Obtaining a static solution requires

$$\Lambda = B^2/2 \quad (4)$$

which is the usual cosmological constant problem. The radius of the sphere is

$$r = (\kappa B)^{-1} \quad (5)$$

where  $\kappa^2$  is the six-dimensional Newton's constant. What is intriguing about this model is that if we add a brane at the north and south poles ( $\theta = \theta^i$ ), each with tension  $f^4$ , corresponding to a stress tensor

$$T_{\mu\nu} = -g_{\mu\nu} \frac{f^4}{2\pi} \frac{\sqrt{g^{(4)}}}{\sqrt{G^{(6)}}} \sum_i \delta(\theta - \theta^i), \quad (6)$$

a new static solution is obtained without any additional fine tuning. The new geometry is still locally a sphere, but now there is a deficit angle

$$\gamma = \kappa^2 f^4 \quad (7)$$

corresponding to a local change in curvature at the locations of the branes,

$$R_{ij} = g_{ij} + g_{ij} \frac{\gamma}{\beta} \sum_i \frac{\delta(\theta - \theta^i)}{2\pi \sin \theta}. \quad (8)$$

In terms of the parameters above,

$$\beta = 1 - \frac{\gamma}{2\pi}. \quad (9)$$

It is transparent that the geometry is locally still a sphere, since we may rescale  $\alpha$  so that

$$ds^2 = r^2[d\theta^2 + \sin^2 \theta d\phi^2] \quad (10)$$

but now  $0 \leq \phi \leq 2\pi\beta$ . This coordinate system is a physically convenient choice for the linear perturbation analysis, since all the effect of the tension and deficit angle is put into the boundary.

With a non-vanishing tension the area of the sphere is smaller due to the deficit angle. As a result, the magnetic flux depends on the tension. If a quantization condition is imposed on the flux, then the self tuning feature of this model would no longer work, for then the cancellation of bulk parameters needed to obtain a static solution would depend on the tension [16].

To avoid this conclusion we will assume that there is no quantization condition. This requires us to assume that there are no electric sources for the bulk  $U(1)$  gauge field<sup>3</sup>. Since the Standard Model fields are not charged under this gauge group, this is not necessarily a phenomenological problem.

### 3 Linear Analysis

#### 3.1 Gauge Fixing and Perturbed Equations

A general perturbation of the background metric (10) is given by

$$\begin{aligned} ds^2 = & (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + 2h_{\mu\theta}dx^\mu d\theta + 2h_{\mu\phi}dx^\mu d\phi \\ & + (1 + h_{\theta\theta})d\theta^2 + 2h_{\theta\phi}d\theta d\phi + \sin^2 \theta(1 + \tilde{h}_{\phi\phi})d\phi^2. \end{aligned} \quad (11)$$

Here we work in units with  $r = 1$ . For non-zero tension there are curvature singularities at  $\theta = 0$  and  $\pi$ . Then  $0 \leq \phi \leq 2\pi\beta$ . When the tension vanishes, there are coordinate singularities at these points.

---

<sup>3</sup>This assumption does not affect the stability analysis.

A convenient gauge choice would be Gaussian Normal-like gauge conditions in the bulk. These are

$$h_{\theta\mu} = h_{\theta\theta} = \partial^\mu h_{\mu\phi} = 0 , \quad (12)$$

providing six conditions.

There is an important subtlety in choosing this gauge though, which we now discuss. In the end we will choose a gauge that is very similar to this one.

To set  $h_{\theta\theta}^{new} = 0$  one chooses a gauge parameter

$$\epsilon_\theta(x, \theta) = -\frac{1}{2} \int_0^\theta d\theta' h_{\theta\theta}^{old}(x, \theta') . \quad (13)$$

The problem with this gauge transformation is with the location of the boundaries. In the gauge with  $h_{\theta\theta} = 0$ , the brane at the north pole is located at  $\theta = 0$ , but the brane at the south pole is now located at  $\bar{\theta} = \pi + \epsilon_\theta(x, \pi)$ , or

$$\bar{\theta} = \pi - F(x)\pi/2 \quad (14)$$

in general ( $F$  is defined by these two equations). Imposing boundary conditions at the location of the south pole brane is technically too subtle in this gauge.

Since  $F$  represents a perturbation that cannot be gauged away, it is more convenient to put it in the metric rather than the location of the boundary. This is done by choosing a slightly different gauge parameter,

$$\epsilon_\theta(x, \theta) = -\frac{1}{2} \int_0^\theta d\theta' h_{\theta\theta}^{old}(x, \theta') + \theta F(x)/2 . \quad (15)$$

which keeps the branes located at 0 and  $\pi$ . The price we pay is that we cannot completely gauge  $h_{\theta\theta}$  away, for in this gauge  $h_{\theta\theta}^{new} = F(x)$ .

We use the  $U(1)$  gauge invariance to set

$$\partial^\mu a_\mu = 0 . \quad (16)$$

This together with

$$h_{\theta\mu} = \partial^\mu h_{\mu\phi} = 0 \quad (17)$$

and

$$h_{\theta\theta}(\theta, x) = F(x) \quad (18)$$

represent our gauge conditions.

We search for massless scalar perturbations only, since only these lead to possibly dangerous long distance deviations from Einstein gravity.

In addition, we focus on scalar perturbations that are independent of the angular coordinate  $\phi$ . The reason is that only  $\phi$ -independent scalar perturbations couple to matter on the brane, either through kinetic mixing with the graviton or because of a non-vanishing wavefunction at the brane locations. Perturbations with non-trivial  $\phi$  dependence only “see” the brane tension through a change in their periodicity, and will therefore still vanish at the location of the branes.

All this gauge fixing leaves seven scalar perturbations,

$$ds^2 = \left( \eta_{\mu\nu} + \partial_\mu \partial_\nu \lambda + \frac{1}{4} \eta_{\mu\nu} h_{(4)} \right) dx^\mu dx^\nu + (1 + F) d\theta^2 + 2h_{\theta\phi} d\theta d\phi + \sin^2 \theta (1 + \tilde{h}_{\phi\phi}) d\phi^2 \quad (19)$$

and  $A = a_\theta d\theta + a_\phi d\phi$  with field strengths  $f_{AB} = \partial_A a_B - \partial_B a_A$ .

Next Einstein’s equations

$$E_{AB} = \kappa^2 T_{AB} \quad (20)$$

in units with  $r_0 = 1$  are obtained. Focusing on  $\phi$ -independent perturbations, the linearized Einstein equations in the bulk and in the gauges (16-18) are given by

$$\begin{aligned} (\mu\nu) : 0 &= -\frac{1}{2}(\square + \partial_\theta^2 + \cot \theta \partial_\theta) (h_{\mu\nu} - \eta_{\mu\nu} h_{(4)}) - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \partial^\sigma h_{\rho\sigma} \\ &+ \frac{1}{2} \eta_{\mu\nu} (\square \tilde{h}_{\phi\phi} + \square F + F - \tilde{h}_{\phi\phi}) + \frac{1}{2} (\partial_\mu \partial_\rho h_\nu^\rho + (\nu \leftrightarrow \mu)) \\ &+ \frac{1}{2} \eta_{\mu\nu} \left( \partial_\theta^2 \tilde{h}_{\phi\phi} + 2 \cot \theta \partial_\theta \tilde{h}_{\phi\phi} \right) \\ &- \frac{1}{2} \partial_\mu \partial_\nu h_{(4)} - \frac{1}{2} \partial_\mu \partial_\nu \tilde{h}_{\phi\phi} - \frac{1}{2} \partial_\mu \partial_\nu F + \eta_{\mu\nu} \frac{\kappa^2 B}{\sin \theta} \partial_\theta a_\phi \end{aligned} \quad (21)$$

$$(\theta\mu) : 0 = \partial_\theta \partial^\rho h_{\rho\mu} + \partial_\mu \left( \cot \theta (F - \tilde{h}_{\phi\phi}) - \partial_\theta h_{(4)} - \partial_\theta \tilde{h}_{\phi\phi} - 2\kappa^2 B \frac{a_\phi}{\sin \theta} \right) \quad (22)$$

$$\begin{aligned} (\phi\mu) : 0 &= -(\square + \partial_\theta^2 - \cot \theta \partial_\theta) h_{\phi\mu} \\ &+ \partial_\mu (\partial_\theta h_{\theta\phi} + \cot \theta h_{\theta\phi} + 2\kappa^2 B \sin \theta a_\theta) - 2\kappa^2 B \sin \theta \partial_\theta a_\mu \end{aligned} \quad (23)$$

$$\begin{aligned} (\theta\theta) : 0 &= \square h_{(4)} - \partial^\mu \partial^\nu h_{\mu\nu} + \square \tilde{h}_{\phi\phi} + \cot \theta \partial_\theta h_{(4)} + F + \tilde{h}_{\phi\phi} \\ &- 2\kappa^2 B \frac{\partial_\theta a_\phi}{\sin \theta} \end{aligned} \quad (24)$$

$$(\theta\theta - \phi\phi) : 0 = \partial_\theta^2 h_{(4)} - \cot \theta \partial_\theta h_{(4)} + \square F - \square \tilde{h}_{\phi\phi} \quad (25)$$

$$(\theta\phi) : 0 = -\square h_{\theta\phi} . \quad (26)$$

The linearized  $U(1)$  gauge equations in these gauges are

$$(\mu) : 0 = \left( \square_{(4)} + \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right) a_\mu - \partial_\mu (\nabla \cdot a) - \frac{B}{\sin \theta} \partial_\theta h_{\mu\phi} \quad (27)$$

$$(\theta) : 0 = \square_{(4)} a_\theta \quad (28)$$

$$(\phi) : 0 = \sin \theta \partial_\theta \left( \frac{1}{\sin \theta} \partial_\theta a_\phi \right) + \square a_\phi - \frac{B}{2} \sin \theta (\partial_\theta \tilde{h}_{\phi\phi} - \partial_\theta h_{(4)}) \quad (29)$$

with  $\nabla \cdot a = \partial_\theta a_\theta + \cot \theta a_\theta$  in the Lorentz gauge and acting on  $\phi$  independent perturbations, and  $\square = \partial^\mu \partial_\mu$ . For massive states, (27) together with the gauge choices (16) and (17) imply  $\nabla \cdot a = 0$ . These equations in the bulk must be supplemented with boundary conditions imposed at the locations of the conical singularities. Boundary conditions are discussed in the next section.

We emphasize that the equations obtained above are valid for arbitrary brane tension. Due to the choice of parameterization of the background given by (10), the effect of the brane tension appears only in the boundary conditions, and for solutions with non-trivial  $\phi$  dependence (which we are not looking at here), in  $2\pi\beta$  periodicity conditions.

As may be anticipated by considering the four-dimensional effective potential [4, 17], the radion in this model is massive. It is given by the mode

$$h_{\theta\theta} = \tilde{h}_{\phi\phi} = -\frac{h_{(4)}}{4} = F(x) , \quad (30)$$

with all other fields vanishing. It has a mass  $m^2 = 1/r_0^2$ . This result agrees with previous computations [15].

In the remainder of this note we focus on massless modes only.

For the zero modes there is in addition the usual residual gauge invariances  $\Omega$  where  $\square \Omega = 0$ . Here we have a residual  $U(1)$  gauge invariance  $\Lambda$ , and residual diffeomorphism invariances  $\epsilon_\phi$  and  $\epsilon_\mu$  only, since  $\epsilon_\theta$  is fixed by our gauge choice. We use  $\Lambda$  and  $\epsilon_\phi$  to set the zero modes of  $a_\mu$  and  $h_{\phi\mu}$  to be purely transverse, and  $\epsilon_\mu$  to set the vector components of the four-dimensional zero mode graviton to zero.

In total there are naively 10 equations for 7 variables. But fortunately not all of these equations are independent. Equation (29) and the trace of (22) are derivable from other equations. Equations (26) and (28) are trivial acting on the zero modes we are focusing on. This leaves six non-trivial equations for seven scalar perturbations (but recall that  $F$  is pure gauge in the bulk).

Inspecting the equations further,  $a_\theta$  and  $h_{\theta\phi}$  decouple from the other perturbations. Their wavefunctions are fixed by (23) and (27) to be

$$a_\theta = \frac{c_0(x)}{\sin \theta} , \quad h_{\theta\phi} = 2\kappa^2 B c_0(x) \cot \theta . \quad (31)$$

Next we proceed to solving the other equations. Equation (25) can be solved immediately to give

$$h_{(4)} = c_1(x) - c_2(x) \cos \theta . \quad (32)$$

Next use (24) to solve for  $\tilde{h}_{\phi\phi}$  and substitute this into (22) to obtain

$$\frac{\partial^2 a_\phi}{\partial \theta^2} + a_\phi = -\frac{c_2(x)}{8\kappa^2 B} (7 - 11 \cos^2 \theta) + \frac{F(x)}{\kappa^2 B} \cos \theta \quad (33)$$

This will have two homogeneous solutions and one homogeneous solution. Since the sources are independent, we may think of this as four solutions in total. Finally, (22) with  $\mu \neq \nu$  determines  $\lambda$  in terms of the previous solutions.

The most general solutions to these equations are given by

$$\begin{aligned} \tilde{h}_{\phi\phi} &= c_3(x) + \frac{5}{6} c_2(x) \cos \theta + 2\kappa^2 B c_4(x) \cot \theta + \theta \cot \theta F(x) \\ a_\phi &= -\frac{c_3(x)}{2\kappa^2 B} \cos \theta + \frac{c_2(x)}{24\kappa^2 B} (1 - 11 \cos^2 \theta) + c_4(x) \sin \theta \\ &\quad + \frac{F(x)}{2\kappa^2 B} \theta \sin \theta \\ h_{\theta\theta} &= F(x) \end{aligned} \quad (34)$$

together with (32).

To summarize, for an arbitrary tension we have found the the most general  $\phi$ -independent massless scalar perturbation solution to the  $U(1)$  and Einstein field equations. Imposing boundary conditions on these solutions is discussed in the next section.

### 3.2 Boundary Conditions

For non-zero tension the locations of the branes are special points on the sphere with curvature singularities. To obtain the boundary conditions for the perturbations at these points we need to inspect the field equations and match the singularities.

It is useful to rewrite the internal metric, including perturbations, as

$$ds_{int}^2 = r_0^2 (1 + h_{\theta\theta}) \left[ d\theta^2 + \beta^2 \sin^2 \theta (1 + \tilde{h}_{\phi\phi} - h_{\theta\theta}) d\alpha^2 \right] , \quad (35)$$

where in a change of notation,  $r_0 = (\kappa B)^{-1}$  denotes the unperturbed radius. To linear order in the perturbations this is equivalent to our previous parameterization. Here we are also only focusing on  $h_{\theta\theta}$  and  $\tilde{h}_{\phi\phi}$ , since  $h_{\theta\phi}$  decouples and does not couple to brane matter. For long-wavelength perturbations this describes a new background with effective radius

$$r^2 = r_0^2 (1 + h_{\theta\theta}) \quad (36)$$

and effective deficit angle

$$\tilde{\beta}^2 = \beta^2(1 + \tilde{h}_{\phi\phi} - h_{\theta\theta}) . \quad (37)$$

To obtain first order terms in the equations of motion with explicit delta-function singularities, which correspond to perturbations without derivatives, we may use (8), using (9) and substituting for the effective radius (36) and effective deficit angle (37).

This implies that to linear order there are no singular contributions to the  $(i, j)$  Einstein equations. There is however, a singular contribution to the  $(\mu, \nu)$  equation, given by

$$\begin{aligned} E_{\mu\nu}|_{sing.} &= \sum_i g_{\mu\nu} \frac{1}{\tilde{\beta}r^2} [\tilde{\beta} - 1] \frac{\delta(\theta - \theta^i)}{\sin \theta} \\ &= \kappa^2 T_{\mu\nu}|_{sing.} \\ &= -\kappa^2 f^4 \sum_i g_{\mu\nu} \frac{\sqrt{g^{(4)}}}{2\pi\sqrt{G^{(6)}}} \delta(\theta - \theta^i) = -\kappa^2 f^4 \sum_i g_{\mu\nu} \frac{1}{\tilde{\beta}r^2} \frac{\delta(\theta - \theta^i)}{2\pi \sin \theta} \end{aligned} \quad (38)$$

Before concluding that these singularities must match, we must check that perturbations involving derivatives cannot contribute additional singularities. To see that they cannot, use the  $(\mu\theta)$  equation to solve for  $\tilde{h}_{\phi\phi}$  and substitute it into the  $(\mu = \nu)$  equation. One finds all the terms cancel identically, except for the singularities appearing in (38). Inspecting (38), we see that to satisfy the equations of motion requires

$$[\tilde{\beta} - 1] = -\frac{\kappa^2}{2\pi} f^4 \quad (39)$$

or

$$[\tilde{h}_{\phi\phi} - h_{\theta\theta}] = 0 \quad (40)$$

at the location of either brane. This is just the statement that without a change in the tension, a perturbation in the metric cannot change the deficit angle.

To obtain other boundary conditions we must inspect the other field equations.

The  $(\theta\theta - \phi\phi)$  metric equation does not contain any curvature singularities. Requiring that the solution is finite gives the boundary condition

$$\partial_\theta h_{(4)}| = 0 . \quad (41)$$

Finally, to obtain the boundary condition for  $a_\theta$ , consider spreading the brane out into a ring located at  $\theta \sim \epsilon$ . Later we will send  $\epsilon \rightarrow 0$ . Inside the ring we have no deficit angle, so all fields are regular at the pole, and therefore  $a_\theta \rightarrow 0$ . The ring does not affect the  $a_\mu$  gauge equation of motion, so  $\nabla \cdot a = 0$  is the  $a_\theta$  equation both inside and

outside the ring. The only solution inside the ring, consistent with this equation and the boundary condition at the pole, is  $a_\theta = 0$ . By continuity at the location of the ring, the boundary condition outside the ring is then

$$a_\theta| = 0 . \quad (42)$$

Equations (40), (41) and (42) are our boundary conditions when the tension is non-vanishing.

For vanishing tension, there are stronger constraints than these from requiring that the north and south poles not be special points. This means that our solutions in polar coordinates should have sensible (C) values when expressed in a Cartesian basis local to the poles. This implies that near a pole,

$$h_{\mu\phi} \sim f_{\mu\phi} \sim \tilde{h}_{\phi\phi} - h_{\theta\theta} \sim \theta^2 \quad (43)$$

Inspecting the four solutions above, we first note that the normal mode  $c_0(x)$  and  $c_4(x)$  are too singular to satisfy the boundary conditions. However the linear combination

$$c_3(x) = -\frac{5}{6}c_2(x) \quad (44)$$

and the normal mode  $F(x)$  both independently satisfy the boundary conditions at  $\theta = 0$ . Thus there are two independent solutions that satisfy the boundary conditions at  $\theta = 0$ .

But neither solution satisfies the boundary conditions at  $\theta = \pi$ . As  $\theta \rightarrow \pi$ , a general combination of these two solutions behaves as

$$\tilde{h}_{\phi\phi} - h_{\theta\theta} \rightarrow -\frac{5}{3}c_2(x) - F(x) - \frac{\pi}{\sin \theta}F(x) . \quad (45)$$

The boundary condition implies that the left side should vanish at the boundary, so both  $c_2 = F = 0$ . Thus there are no solutions that satisfy the boundary conditions at both  $\theta = 0$  and  $\theta = \pi$ .

This conclusion is true for arbitrary tension. It is straightforward to repeat the analysis for vanishing tension. The only difference is that the boundary conditions are *stronger*, because of the constraint of regularity. No zero modes therefore exist in this limit either. Thus any light mode with mass that vanishes as the tension is sent to zero is also excluded by our analysis.

## 4 Fine-tuning or Self-tuning?

We return to an issue briefly raised in the introduction. One might think that the relation (7) between the deficit angle and the brane tension does not represent a

fine-tuning, since the deficit angle is an integration parameter, not a parameter of the Lagrangian. But a simple four-dimensional counter-example illustrates that the issue is not as straightforward [18].

Consider a four dimensional theory with a bare cosmological constant  $\Lambda_0$  and a four-form field strength with value

$$F_{\mu\nu\rho\sigma} = c\epsilon_{\mu\nu\rho\sigma} \quad (46)$$

which satisfies the field equations of motion and where  $c$  is an integration parameter [19]. The source for gravity in this theory is

$$\Lambda = \Lambda_0 + c^2. \quad (47)$$

The integration parameter may be chosen to be  $c^2 = -\Lambda_0$ , giving a flat space solution. But obviously this is not the only solution, as there is a family of de-Sitter and Anti-de-Sitter solutions.

In analogy with this four-form example, does setting the deficit angle - an integration parameter - of the six-dimensional model to be equal to the tension involve a fine-tuning? For recall that it has not been demonstrated that the deficit angle is forced by the equations of motion to be equal to the tension—that only followed from the equations of motion after assuming a flat-space ansatz. For maximally symmetric space-times, [15] has found that once the bulk cosmological constant is finetuned against the magnetic flux, then de-Sitter or Anti-de Sitter solutions along the brane directions are forbidden, and the deficit angle is equal to the tension.

But that still does not completely address the issue. In other words, it isn't clear that a change in the tension is canceled by a change in the deficit angle. If other cosmological solutions exist, for the same tension but other values of the deficit angle, then (7) is a fine-tuning, and this model would then be less appealing.

What might these solutions look like? <sup>4</sup> A dynamical change in the tension could lead to a solution that is not maximally symmetric, but instead interpolates between an inflating or generally time-dependent solution near the brane, to a static geometry far from the brane. Far from the brane, the deficit angle would be (approximately) equal to its unperturbed value.

## 5 Conclusions

We have performed a linear perturbation analysis of the model presented in [6] and [7] to search for phenomenologically dangerous massless or approximately massless scalars. After imposing the boundary conditions, we have found that there are no such modes.

---

<sup>4</sup>The authors thank Maxim Perelstein for discussions on this point.

If this model does have a self-tuning mechanism, then our results do raise a puzzle. Namely, below the compactification scale the only light modes are the four dimensional graviton, the Standard Model fields, a gravi-vector boson from the residual isometry of the bulk, and a  $U(1)$  gauge boson. The latter two do not couple to the tension on the brane and so we can forget about them. The puzzle is that one might have expected that the self-tuning of a small enough change in the brane tension could be understood in the four dimensional effective theory—but this does not seem likely, since our results show that this model lacks any additional light degree of freedom.

If this is the case here, then in order for this model to be compatible with the observed size or bound on the cosmological constant, we would need  $1/r \sim 10^{-3}$  eV [5]. There is an additional puzzle here, if the model does self-tune: from the higher dimensional perspective, it appears that it is the full quantum mechanical tension on the brane that is canceled, not just the high energy contribution. If this were true, then there would not be a constraint on the size of the internal space arising from vacuum fluctuations of brane localised matter.

Our results suggest that if there is self-tuning, then it must be due to modes no lighter than the compactification scale. A consistent story would then be that the massive states only partially cancel a dynamical change in the tension, down to an amount set by the compactification scale.

Finally, demonstrating that these models do or do not self-tune might be difficult: one would need to find or demonstrate the absence of non-maximally symmetric solutions with an approximately static deficit angle and geometry far from the brane, but with a cosmological geometry near the brane. That is, to follow the cosmological evolution of this system through a phase transition.

## Acknowledgments

We would like to thank K. Akama, C. Burgess, C. Csaki, M. Giovannini, C. Grojean, M. Luty, M. Perelstein, J. Vinet and M. Wise for discussions and comments. This work is supported by the U.S. Department of Energy under contract number DE-FG-0392-ER40701.

## References

[1] N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys.Lett.**B429**:263-272,1998, hep-ph/9803315;I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali,

Phys.Lett.**B436**:257-263,1998, hep-ph/9804398; N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys.Rev.**D59**:086004,1999, hep-ph/9807344; L. Randall, R. Sundrum, Phys.Rev.Lett.**83**:3370-3373,1999, hep-ph/9905221; L. Randall, R. Sundrum, Phys.Rev.Lett.**83**:4690-4693,1999, hep-th/9906064.

[2] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper, R. Sundrum, Phys.Lett.**B480**:193-199,2000, hep-th/0001197 ; S. Kachru, E. Silverstein, M. Schulz, Phys.Rev.**D62**:045021,2000, hep-th/0001206.

[3] K. Akama, "Pregeometry" in Lecture Notes in Physics, 176, Gauge Theory and Gravitation, Proceedings, Nara, 1982, edited by K. Kikkawa, N. Nakanishi and H. Nariai, 267-271 (Springer-Verlag,1983), hep-th/0001113; V.A. Rubakov, M.E. Shaposhnikov, Phys.Lett.**B125**:139,1983; V.A. Rubakov, M.E. Shaposhnikov, Phys.Lett.**B125** :136-138,1983.

[4] R. Sundrum, Phys.Rev.**D59**:085009,1999, hep-ph/9805471; Phys.Rev.**D59**:085010,1999, hep-ph/9807348.

[5] J-W. Chen, M. A. Luty, E. Ponton, JHEP 0009:012,2000, hep-th/0003067 .

[6] S.M.Carroll and M.M.Guica, [hep-th/0302067]

[7] I. Navarro, JCAP **0309**:004,2003, hep-th/0302129.

[8] C. Csaki, J. Erlich, C. Grojean, T. J. Hollowood, Nucl.Phys.**B584**:359-386,2000, hep-th/0004133; P. Binetruy, J.M. Cline, C. Grojean, Phys.Lett.**B489**:403-410,2000, hep-th/0007029.

[9] S. Forste, Z. Lalak, S. Lavignac, H. P. Nilles, Phys.Lett.**B481**:360-364,2000, hep-th/0002164.

[10] P. Bostock, R. Gregory, I. Navarro, J. Santiago, hep-th/0311074.

[11] M. Giovannini, Phys.Rev.**D66**:044016,2002 ; M. Giovannini, J.-V. Le Be, S. Riederer Class.Quant.Grav.**19**:3357,2002.

[12] K. Hagiwara *et al.*, Phys. Rev. D **66**, 010001-1 (2002)

[13] K. Nordtvedt, Phys. Rev. **169**,1014,1968; K. Nordtvedt, Phys. Rev. **169**,1017,1968; K. Nordtvedt, Phys. Rev. **170**,1186,1968.

[14] H-P. Nilles, A. Papazoglou, G. Tasinato, Nucl.Phys.B677:405-429,2004, hep-th/0309042.

[15] J.M.Cline, J.Descheneau, M.Giovannini and J.Vinet, JHEP **0306**:048,2003, [hep-th/0304147].

- [16] Y. Aghababaie, C.P. Burgess, S.L. Parameswaran, F. Quevedo, Nucl. Phys. **B680**:389-414, 2004, hep-th/0304256; I. Navarro, Class. Quant. Grav. **20**:3603-3612, 2003, hep-th/0305014; Hyun Min Lee, hep-th/0309050.
- [17] S. M. Carroll, J. Geddes, M. B. Hoffman, R. M. Wald, Phys. Rev. **D66**:024036, 2002, hep-th/0110149 .
- [18] S. Weinberg, Rev. Mod. Phys. **61**:1-23, 1989.
- [19] E. Witten, in *Proceedings of the 1983 Shelter Island Conference on Quantum Field Theory and the Fundamental Problems of Physics*, edited by R. Jackiw, N.N. Khuri, S. Weinberg, and E. Witten (MIT, Cambridge, Massachusetts), 1983; M. Henneaux and C. Teitelboim, Phys. Lett. **B143**:415, 1984.